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Quantum mechanics via extended measures

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Abstract. Quantum phenomena are discussed using extended measures. The complex measurable processes introduced earlier are studied in greater depth in the context of harmonic oscillators. Some of the fundamental problems of radiation and interaction of radiation with matter are analysed in the general frame of extended measure theory. The connection to Feynman path integrals is also discussed and it is shown how the present approach circumvents the divergence difficulties particularly in the estimation of Lamb shift.

1. Introduction

The object of this contribution is to explore the possibility of a formulation of quantum phenomena using extended measures and measurable processes (Srinivasan and Sudarshan 1994, 1996). The motivation to do so stems from the basic problems that arise in the interpretation of the quantum mechanics as practised at the current time. No doubt the predictions of quantum mechanics have been very successful in the sense that good agreement has been obtained between theory and experiment; although the method of calculation is a little bit questionable (see Dirac 1978), the agreement and the level of accuracy, particularly in quantum electrodynamics, baffle everyone. However, basic questions involving wavefunction collapse, measurement and the role of observer continue to be as puzzling as they were at the beginning of this century. An unfortunate aspect of the very formulation of quantum mechanics is the secondary role assigned to probability, despite the fact that uncertainty was the symptom to be observed and taken into account in the early stage of development. Thus the whole emphasis all along has been (see, for example, Von Neumann (1955)) on the Hilbert space and operators defined on it with a minor role assigned to probability through Born interpretation. This was partly remedied by the Feynman path integral approach (Feynman 1948, 1950) that provides an attractive formulation of quantum mechanics and relativistic quantum field theory. In fact the path integral theory translates classical time evolution in quantum mechanical terms in a simple and direct manner and provides a description in terms of trajectories, each of which contributes to the quantum mechanical complex amplitude. In a certain sense, it provides a completely independent and self-contained formulation of quantum mechanics and is capable of direct extension to cover gauge fields and quantum gravity (see Fadeev and Slavnov 1980). To bring home the idea that the path integral theory is a completely independent and self-contained formulation of quantum mechanics, Feynman and Hibbs (1965) have outlined how quantum mechanics can be rebuilt from the early work of Dirac (1933); in the same spirit the quantum theory of the electromagnetic field and the deeper problem of interaction of the field with matter had

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been analysed in all its details. The idea of summation over paths is especially appealing to mathematicians and in spite of an early set back in the problem of the identification of the path integral with complex measure (see Cameron 1960), there has been continuing interest in the study of the path integral formalism (Nelson 1964, Cameron 1962). However, it is rather surprising that, in spite of these developments, no attempt had been made until very recently to study the precise implication of the use of the path integral formalism (PIF). The work of Gellman and Hartle (1993) in a certain sense clarifies some of the aspects; the recent work of Youssef (1991, 1994, 1995) inspired by the Bayesian approach and the axioms of Cox (1946) attempts to accommodate the PIF within the framework of conditional complex probability distributions. Youssef (1991) has an elaborate set of axioms more or less on the lines originally proposed by Von Neumann (1933); however, he also assigns a minor role to the process generated by the complex probability. Nevertheless, topics such as interference and mixtures of states are treated in a rather serious manner and the viability of complex probability to describe mixtures of states is established. Complex probability theory is also shown to be consistent with Bell's theorem and other limitations on local realistic theories that more or less agree with predictions of quantum mechanics. However, there is a great need for the interpretation of complex probability or complex measure theory as such. It is precisely in this context that the complex measure theoretic framework (CMTF) was used in paper I (Srinivasan and Sudarshan 1994) to generate stochastic processes in the extended sense. Some general constraints imposed at a local level give rise to a more general structure than the Schrödinger structure; further constraints lead to a Fokker–Planck type of equation for the conditional measure density. We wish to pursue this line of development and look at very specific problems. We also try to identify the Feynman path integral and the so-called measure associated with it. In summary, instead of concluding that classical ideas combined with elementary notions of probability fail to explain the two-slit experiment, we wish to explore the possibility of an extended measure structure and measurable processes arising therefrom forming the basis for the evolution of the physical system. In such a formulation, as there will be a direct role for the generalized probability as such, a distinct advantage may accrue in dealing with models of physical phenomena.

The layout of the paper is as follows. In section 2 we briefly review some of the basic results of the CMTF and present further new results in the same direction. We then proceed to discuss in section 3 a model of a harmonic oscillator which has all the properties of a conventional quantum oscillator. We also show the viability of the model in accommodating coherent states and related phenomena. In section 4 we analyse the problem of the forced harmonic oscillator and link it to the problem of interaction of radiation with matter which was dealt with by Feynman. While the linkage enables us to deal with practical problems of quantum mechanics, the CMTF enables us to interpret the results in all the intermediary stages probabilistically (in a complex measure theoretic sense); the divergence difficulties disappear and some of the earlier calculations in the literature are thereby validated. The final section contains a short discussion and summary of the results.

2. Extended measures and measurable processes

In earlier contributions (Srinivasan and Sudarshan (1994), Srinivasan (1995) referred to as papers I and II), complex measurable processes were introduced and some of their interesting characteristics deduced; in particular, it was shown that the CMTF is viable enough to describe some of the basic physical phenomena at a quantum level. Moreover, the complex measure of sample paths of processes provides a link to the heuristic Feynman way of summation over paths. While the CMTF maintains its identity and distinctiveness from the PIF, the probabilistic basis enables us to tackle problems that cannot be handled effectively by PIF. To enable a smooth passage to the discussion on the harmonic oscillator and associated problems we present a brief survey of the results presented in paper I. We also take the opportunity to discuss a few new results that have a direct bearing on the discussion to follow.

Complex measure is defined as an extension of the concept of signed measure (Ash 1972): for any set $A \in \mathcal{B}$ where $\mathcal{B}$ is a σ-ring of subsets of Ω and $(\Omega, \mathcal{B})$ is a measurable space, we can define the complex measure μ by

$$\mu(A) = \mu_1(A) + \iota\mu_2(A) \tag{2.1}$$

where μ_1 and μ_2 are any two finite signed measures on $(\Omega, \mathcal{B})$. For convenience we use the constraints, for $\forall A \in \mathcal{B}$,

$$\mu(\Omega) = 1 \qquad |\mu(A)| < \infty. \tag{2.2}$$

We follow Pitt (1963) for the introduction of the random variable and set $\Omega = R$; thus we talk of a random integer if Ω is the set of integers, a random real variable when Ω is the set of reals and a random vector when $\Omega = R^k$. We next introduce the notion of a general random sequence (discrete stochastic process) or a general random function (arbitrary stochastic process). To do this we observe that the definition of a general random vector needs to be extended. This is best done by adapting the method due to Pitt (1963). We skip the details and summarize the conclusion by observing that it is indeed possible to deal with an indexed family of complex measures in a consistent way irrespective of whether the index set is discrete or a continuum; such an indexed family is called a complex measurable (stochastic) process, the existence of a measure space over which a stochastic process is defined following on lines parallel to those adopted by Pitt. Such an indexed family of complex measures is best defined by the conditional distributions. From now on we use the symbol Pr to denote the complex measure

$$\begin{aligned} F_n(x, t|x_0,\ t_0; x_1, t_1; \ldots; x_n, t_n) &= \Pr\{X(t) \leqslant x | X(t_0) = x_0, X(t_1) = x_1, \ldots, X(t_n) = x_n\} \\ t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t &\qquad n = 0, 1, 2, \ldots \end{aligned} \tag{2.3}$$

or the conditional density functions (whenever they are defined)

$$\begin{aligned} f_n(x, t|x_0, t_0; x_1, t_1; \ldots; x_n, t_n) &= \lim_{\Delta \to 0} \Pr\{x < X(t) < x + \Delta | X(t_0) \\ &= x_0, X(t_1) = x_1, \ldots; X(t_n) = x_n\}/\Delta. \end{aligned} \tag{2.4}$$

The notion of the Markov property and stationarity can be carried *in toto* from the standard theory of stochastic processes; a process is called Markov if

$$F_n(x, t|x_0, t_0; x_1, t_1; \ldots; x_n, t_n) = F_2(x, t|x_n, t_n) \tag{2.5}$$

or

$$f_n(x, t|x_0, t_0; x_1, t_1; \ldots; x_n, t_n) = f_2(x, t|x_n, t_n) \tag{2.6}$$

for every choice of $x, x_0, x_1, \ldots, x_n$ and $t_0, t_1, t_2, \ldots, t_n$ subject to $t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$.

Some of the interesting results that are of importance in the case of non-negative measures have their counterparts in our present case. Here we discuss the Fokker–Planck equation and the random walk problem in some detail. If we assume the Markov property and time homogeneity, the moment functions can be defined by

$$a_n(z, \Delta) = \int (x - z)^n f_2(x, \Delta|z)\, \mathrm{d}x. \tag{2.7}$$

If at this stage we represent the constraints on f_2 by

$$\lim_{\Delta\to 0}\frac{a_1(z,\Delta)}{\Delta}=A(z)\qquad \lim_{\Delta\to 0}\frac{a_2(z,\Delta)}{\Delta}=B(z) \tag{2.8}$$

and

$$\lim_{\Delta\to 0}\frac{a_n(z,\Delta)}{\Delta}=0\qquad n>2 \tag{2.9}$$

we obtain the Fokker–Planck equation

$$\frac{\partial f_2(x,t|x_0,t_0)}{\partial t}=-\frac{\partial}{\partial x}[f_2(x,t|x_0,t_0)A(x)]+\frac{1}{2}\frac{\partial^2}{\partial x^2}[f_2(x,t|x_0,t_0)B(x)]. \tag{2.10}$$

It should be noted that although $X(t)$ is a real-valued random variable, the expected values $a_n(z,\Delta)$ are, in general, complex-valued functions and so are $A(x)$ and $B(x)$. If time homogeneity is not assumed, then A and B are functions of t as well and with this understanding (2.10) still holds good.

There are two special cases that are important viewed from the development of quantum mechanics. The first is the motion of the harmonic oscillator and is readily realized by setting

$$A(x)=-\iota\omega x \tag{2.11}$$

and

$$B(x)=2D=\frac{\iota\hbar}{M} \tag{2.12}$$

where the notation in terms of D is suggestive of the diffusion coefficient in the classical context. It is to be noted that CMTF gives us the privilege of the use of complex-valued functions/parameters. The resulting Fokker–Planck equation can be written in the form

$$\iota\hbar\frac{\partial\rho(x,t|x_0,t_0)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\rho(x,t|x_0,t_0)}{\partial x^2}+\frac{1}{2}\rho(x,t|x_0,t_0)(M\omega^2x^2+\hbar\omega) \tag{2.13}$$

where

$$\rho(x,t|x_0,t_0)=f_2(x,t|x_0,t_0)\exp\left(\frac{M\omega}{2\hbar}(x^2-x_0^2)\right). \tag{2.14}$$

If we define

$$\Psi(x,t|x_0,t_0)=\rho(x,t|x_0,t_0)\exp\left(-\frac{\iota\omega t}{2}\right) \tag{2.15}$$

then $\Psi(x,t|x_0,t_0)$ can be identified to be the familiar Schrödinger wavefunction. It is to be especially noted that the Feynman PIF essentially deals with $\Psi(x,t|x_0,t_0)$. Equation (2.10) can be solved for the special choice (2.11) and the solution along with the relations (2.14) and (2.15) will provide the desirable connection to the path integral formula; the stationary characteristics from the view point of CMTF will enable us to obtain the properties of the quantum harmonic oscillator. The multiplying factors implied by (2.14) and (2.15) are quite important and we will return to this point in the next section.

The next special case is obtained from the three-dimensional generalization of (2.10) which is straightforward. We note that the coefficient A arises from the drift and the case we have in mind corresponds to a central potential $V(r)$; thus we take the corresponding coefficient to arise from the radial drift $A(r)$ given by

$$A(r)=\left(\frac{2K-V(r)}{\mu}\right)^{\frac{1}{2}} \tag{2.16}$$

where μ is the (effective) mass and $V(r)$ for a Coulomb type of force is given by

$$V(r) = -\frac{Ze^2}{r}. \tag{2.17}$$

The diffusion $D = \frac{1}{2}B(x)$ is taken to be isotropic and equal to $\frac{1}{2}\iota\overline{h}/\mu$. Then the Fokker–Planck (FP) equation in spherical polar system takes the form

$$\frac{\partial f}{\partial t} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A f) + D\nabla^2 f. \tag{2.18}$$

If we set

$$f = \rho \exp\left(\frac{1}{2D}\int^r \left(\frac{2(K - V(\xi))}{\mu}\right)^{\frac{1}{2}} \mathrm{d}\xi\right) \tag{2.19}$$

then we find

$$\iota\overline{h}\frac{\partial\rho}{\partial t} = -\frac{\overline{h}^2}{2\mu}\nabla^2\rho + \rho\left\{V(r) - K - \frac{\iota\overline{h}}{r}\left(\frac{2}{\mu}(K - V(r))\right)^{1/2} - \frac{1}{2}\frac{\iota\overline{h}V'(r)}{(2\mu(K - V(r))^{1/2}}\right\}. \tag{2.20}$$

If we choose

$$V(r) = -\frac{Ze^2}{r} \qquad K = -\frac{\mu Z^2 e^4}{2\overline{h}^2} \tag{2.21}$$

and set

$$\rho = \Psi \exp\left(\iota\frac{\mu z^2 e^4}{2\overline{h}^3}t\right) \tag{2.22}$$

then equation (2.20) reduces to the Schrödinger model for the hydrogen atom provided we neglect the imaginary part of the potential. The time-dependent factor on the right-hand side of (2.22) is a special feature of the CMTF and is intimately connected to the stationary characteristics of the system, a point that will be discussed in the next section in relation to harmonic oscillators.

We next proceed to deal with the CMTF from a practical point of view. While the two special cases briefly discussed above may bring home the importance of complex measurable processes, there remains the major problem of relating the process to physical measurements. Viewed even from the point of statistical inference, the complex measure must be considerably tamed so that it can be related to down to earth frequency ratios. To achieve this we must make a measure transformation, the new measure being positive definite; in such a process a good part of many of the advantages of the CMTF may be lost. These problems do exist in the conventional formulation of quantum phenomena and we may have to live with them in the CMTF too. It was with this in view that the two measures were proposed in paper I; the first known as the mod measure is defined for any set $A \in \mathcal{B}$ (see Halmos 1950)

$$|\mu|(A) = \sup\left|\int_A f\,\mathrm{d}\mu\right| \tag{2.23}$$

where the supremum is extended over all measurable functions f such that $|f| \leqslant 1$. There is another equivalent way to define the mod measure:

$$|\mu|(A) = \sup\left\{\sum_{i=1}^{n}|\mu(E_i)| : E_1, E_2, \ldots, E_n \text{ form a partition of } A\right\} \tag{2.24}$$

where the supremum is over all partitions and $E_1, E_2, \ldots, E_n$ are measurable sets of the family $\mathcal{B}$. It will be shown in the next section that the mod measure is the appropriate one from the CMTF point of view and will yield all the results obtained through a conventional Born interpretation. However, the process of interpretation of the modulus measure of the measure density automatically leads us to the alternative measure of modulus square introduced rather indirectly in paper I.

That the mod measure defined through (2.23) or (2.24) is the most natural one follows from the following point of view. In paper I, it was shown that a typical random walk process in the CMTF has a characteristic diffusive limit; in this connection it is worth noting that, in classical probability theory, limit theorems are generally a consequence of the Chebycheff inequality. Hence, it is worthwhile to explore whether such an inequality is possible in the CMTF. The answer is in the affirmative and best stated in terms of the observable and inference-friendly modulus measure. We call it the:

Generalized Chebycheff inequality. If X is any real random variable, the complex measure which we denote by the symbol Pr satisfies

$$|\Pr\{|X - E[X]_{11}| > k\sigma_{11}\}| \leqslant \frac{K}{k^2} \tag{2.25}$$

where k is any arbitrary positive number and $E[X]_{11}$ and σ_{11}^2 are respectively the expected value and variance of the random variable X with reference to the mod measure while K is a positive real number characteristic of the measure and given explicitly by

$$K = \int_{-\infty}^{+\infty} |f_1 + \iota f_2| \, \mathrm{d}x \tag{2.26}$$

where $f_1 + \iota f_2$ is the complex measure density of the random variable X.

It is worth noting that the inequality says nothing about the convergence or closeness of the random variable to its expected value. Since observations can be made only on the basis of positive definite probability, the inequality can be called observation friendly. The proof of the inequality is straightforward and follows exactly on lines very similar to the proof of the classical Chebycheff inequality. If the complex measure density is singular and has mass concentrations, the integrand in (2.26) can be interpreted in terms of distributions. In the next section we will have plenty of occasions to use the mod measure and its statistical characteristics.

The complex measure introduced above is viable enough to describe, on the one hand, the motion of oscillators (2.13) (although through the FP structure) and, on the other, the motion of electrons around the nucleus through the Fokker–Planck–Schrödinger equation (2.18)–(2.20) satisfied by the measure density function. To describe particles with internal degree of freedom, we have to continue in the same vein and seek a further extension of measures; (see Srinivasan and Sudarshan (1996), referred to as paper III). Thus it is possible to define the quarternion measure λ for any set $A \in \mathcal{B}$

$$\lambda(A) = \lambda_0(A) + \hat{\imath}\lambda_1(A)\hat{\jmath}\lambda_2(A) + \hat{k}\lambda_3(A) \tag{2.27}$$

where λ_0, λ_1, λ_2 and λ_3 are complex measures defined earlier on $(\Omega, \mathcal{B})$ and $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are hypercomplex numbers introduced by Hamilton (see Smith 1958). Thus we can proceed to introduce an indexed family of such measures and random processes in this extended sense. We can in fact proceed in exactly the same manner as before up to the level of the FP equation. Since our primary motivation is to encompass structures like the Dirac equation within our ambit, we dispense with the apparently unsymmetrical terms like the diffusion arising from the B term in equation (2.10) and attempt to incorporate dispersion through the A term and other means. For instance, in the FP development, the first moment or the expected value of the random variable is now a quaternion and hence a formal Langevin equation does not make much sense unless it is interpreted through the quaternion measure. In paper III it is shown that a quaternion measurable Markov process $X(t)$ fused with another independent two-valued Markov process $Z(t)$ leads to the Dirac equation in the Weyl representation. We outline here the development for the sake of continuity. We note that $X(t)$ is a three-dimensional Markov process and its change may be expressed formally by a Langevin equation of the form

$$\mathrm{d}X^j = V^j \exp[\iota\pi Z(t)]\,\mathrm{d}t \tag{2.28}$$

where $Z(t)$ is a two-valued Markov process on $\{0, \hat{1}\}$ with transition rates $\lambda_+(0 \to 1)$ and $\lambda_-(1 \to 0)$ per unit time and represents the transition from one to the other of the two helicity states. The Langevin equation, although it depicts some sort of dynamics, is rather weak and can be understood only in a formal way because of the lack of specification of the process $X(t)$ itself. A better and perhaps non-controversial way would be to specify the process in terms of the first moment:

$$\begin{aligned} &E[\mathrm{d}X^j|Z(t)=0] = V^j\,\mathrm{d}t + o(\mathrm{d}t)\\ &E[\mathrm{d}X^j|Z(t)=1] = -V^j\,\mathrm{d}t + o(\mathrm{d}t). \end{aligned} \tag{2.29}$$

It should be noted that the expected values are themselves quaternions and the choice (in modern notation of σ-matrices)

$$V^j = c\sigma^j \tag{2.30}$$

brings out clearly the physics of the situation. With a further assumption that higher moments are of smaller order of magnitude compared to $\mathrm{d}t$, the FP method applied to the quaternion measure densities $\pi_\pm(\boldsymbol{x}, t)$ yields

$$\frac{\partial\pi_+(\boldsymbol{x},t)}{\partial t} = -c\sigma\cdot\nabla\pi_+ - (\lambda_+\pi_+ - \lambda_-\pi_-) \tag{2.31}$$

where $\pi_+(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}(\pi_-(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x})$ represents the quaternion measure that $X(t)$ lies in $(\boldsymbol{x}, \boldsymbol{x}+\mathrm{d}\boldsymbol{x})$ and $Z(t) = 0(1)$. Likewise, we have

$$\frac{\partial\pi_-(\boldsymbol{x},t)}{\partial t} = -c\sigma\cdot\nabla\pi_- - (\lambda_-\pi_- - \lambda_+\pi_+). \tag{2.32}$$

If we now post multiply $\pi_\pm$ by an arbitrary two spinor $\chi_\pm$ to yield two component objects and use the same symbol $\pi_\pm$ to denote the resulting two spinors, we obtain, by choosing

$$\lambda_\pm = -\iota\frac{mc^2}{\hbar} \tag{2.33}$$

$$\iota\hbar\frac{\partial\Psi}{\partial t} = mc^2\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{\hbar c}{\iota}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix}\nabla\Psi \tag{2.34}$$

where

$$\Psi = \begin{pmatrix}\pi_+\\ \pi_-\end{pmatrix}\mathrm{e}^{-\iota mc^2 t/\hbar}. \tag{2.35}$$

It is worth noting that plane wave solutions of the form

$$\Psi = u\,\mathrm{e}^{-(\iota/\hbar)(Et-\boldsymbol{p}\cdot\boldsymbol{x})}$$

lead to

$$\begin{pmatrix}\pi_+\\ \pi_-\end{pmatrix} = u\,\mathrm{e}^{-(\iota/\hbar)(E-mc^2)t-\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{2.36}$$

The multiplicative factor introduced in (2.35) is deliberate to bring out the fact that a stationary state in the strict probabilistic sense is obtained if $E = mc^2$.

We next observe that the probabilistic interpretation of the scalar product obtained from the spinor can be understood as a way of introducing a positive definite measure. Encouraged by this we can proceed to interpret the main results and the formalism presented by Feynman and Hibbs (1965). The main emphasis in the CMTF is the probabilistic basis for the PIF. However, it should be conceded that, even in the case of the harmonic oscillator, the CMTF maintains its identity which is apparent from (2.14) and (2.15); the equation satisfied by the complex measure differs from the Schrödinger structure to the extent of a drift term. Even after the elimination of the first-order term which results in a measure transformation that has all the resemblance of a gauge transformation, there is still an additive constant in the potential which characterizes the ground state in a different but probabilistically sound manner. That this type of difference persists for any type of potential is clear from the second model described by equations (2.18) and (2.20). However, an important feature is that we obtain a different complex measure although not normalized. Many of the general techniques developed by Feynman and Hibbs (FH) for the PIF are equally applicable to the measure functions obtained in the CMTF since the major constraint expressed by (2.2) will render it amenable to these techniques by virtue of the measure being absolutely integrable. Thus we can obtain a broad qualitative agreement with most of the results in ch 1–4 of FH. To arrive at the momentum representation we do the following in the CMTF. In view of the strict conditions imposed, the square root of the measure density satisfies the conditions for the Plancharel transform (Feller 1971) and hence we conclude that the Fourier transform also belongs to the L_2 space (generated by such complex measures) which is also a Hilbert space and to that extent we have a reconciliation with the formalism. To make further progress we interpret the results in section 6.5 of FH in the CMTF. Some of the results relating to (conventional Schrödinger) amplitude mechanics can be taken over *in toto*. First we note that for a force free system the CMTF leads to Schrödinger equation. When the potential can be treated as a perturbation, the usual expansion (see, for example, equation (6.68) p 144) has an easy interpretation in the CMTF. Further developments in FH (pp 144–151) can be incorporated in the CMTF on the basis of the mod square measure. Thus the results of ch 5 and 6 can be adapted with trivial modification; in particular, the perturbation treatment can be adopted in most cases where an explicit solution is not available. Even at the risk of repetition, we observe that the CMTF maintains its identity with its versatility of being interpreted as a measure at every stage of calculation.

3. Harmonic oscillators and their applications

We now take up the Fokker–Planck equation describing the evolution of the complex measure density governing the free harmonic oscillator. The general FP equation (2.10) for the specific choice of the diffusion given by (2.11) now takes the form

$$\frac{\partial f_2(x,t|x_0,t_0)}{\partial t} = +\frac{\partial}{\partial x}(\iota\omega x f_2) + \frac{\iota\hbar}{M}\frac{\partial^2 f_2(x,t|x_0,t_0)}{\partial x^2}. \tag{3.1}$$

The specific choice (2.11) implies that the basic stochastic process $\{X(t)\}$ is stationary; thus f_2 is a function of $t - t_0$ and we let $t_0 = 0$ without loss of generality. Equation (3.1) is easily solved by the method of characteristics; using the initial condition

$$f_2(x,0|x_0) = \delta(x - x_0) \tag{3.2}$$

we obtain

$$f_2(x,t|x_0) = \left(\frac{M\omega\,\mathrm{e}^{2\iota\omega t}}{\pi\hbar(\mathrm{e}^{2\iota\omega t}-1)}\right)^{1/2} \exp\left(-\frac{M\omega}{\hbar}\frac{(x\mathrm{e}^{\iota\omega t}-x_0)^2}{\mathrm{e}^{2\iota\omega t}-1}\right). \tag{3.3}$$

At the outset we note that f_2 is a conditional measure density; we make the usual gimmick $\omega \to \omega - \iota\epsilon$ and take the limit as $t \to \infty$. Thus we find

$$\lim_{t\to\infty} t_2(x, t|x_0) = \Pi(x) = \left(\frac{M\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{M\omega x^2}{\hbar}\right). \tag{3.4}$$

The above expression on the right-hand side is indeed the stationary measure density and is a legitimate probability density by virtue of its being positive definite. That $\Pi(x)$ is the stationary density follows by the relation

$$\int f_2(x, t|x_0)\Pi(x_0)\, \mathrm{d}x_0 = \Pi(x). \tag{3.5}$$

The density function $\Pi(x)$ can also be thought of as the modulus measure density by (2.23) or (2.24); in this case (2.25) reduces to the normal Chebycheff inequality with $K = 1$.

To make contact with the literature on harmonic oscillators, we can proceed to solve (3.1) by using the operator formalism and write the solution in terms of the eigenfunctions. Alternatively we take advantage of the solution (3.3) and expand the right-hand side in terms of the natural orthonormal basis of Hermite functions:

$$f_2(x, t|x_0) = \sum \phi_n(x)\phi_n^*(x_0)\, \mathrm{e}^{-(M\omega/2\hbar)(x^2-x_0^2)-\iota n\omega t} \tag{3.6}$$

where

$$\phi_n(x) = (2^n n!)^{-1/2} \left(\frac{M\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{M\omega}{\hbar}}x\right). \tag{3.7}$$

On the other hand, the usual wavefunction $\Psi(x, t|x_0)$ defined by (2.15) admits the expansion

$$\Psi(x, t|x_0) = \sum \phi_n(x)\phi_n^*(x_0)\, \mathrm{e}^{-\iota(n+1/2)\omega t}. \tag{3.8}$$

If we use the connection formula (2.15), we can also obtain from (3.3) a tally with the formula by Feynman by his path integral method:

$$\begin{aligned}\Psi(x, t|x_0) &= f_2(x, t|x_0) \exp\left(\frac{M\omega}{2\hbar}(x^2 - x_0^2) - \iota\frac{\omega t}{2}\right)\\ &= \left(\frac{M\omega}{2\pi\hbar\iota \sin\omega t}\right)^{\frac{1}{2}} \exp\left(\frac{M\iota w}{2\hbar \sin\omega t}((x^2 + x_0^2)\cos\omega t - 2xx_0)\right).\end{aligned} \tag{3.9}$$

At this stage it is worth mentioning that f_2 is a complex measure density properly normalized while Ψ is not. At best it is some kind of a complex measure. The eigenfunction expansion (3.6) may sound a bit odd as compared to (3.8); the reasons are not far to seek. The density function f_2 by virtue of the Markov nature of the underlying complex measurable process satisfies the FP equation (3.1); the FP operator is non-self-adjoint as contrasted with the Schrödinger operator whose eigenfunctions are precisely Hermite functions defined by (3.7). This is a special characteristic of any classical stochastic process (for a discussion see Risken (1984)).

The eigenfunctions of the FP operator are given by

$$\chi_n(x) = \phi_n(x)\mathrm{e}^{-(M\omega/2\hbar)x^2} = (2^n n!)^{-\frac{1}{2}} \left(\frac{M\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{M\omega}{\hbar}}x\right) \mathrm{e}^{-M\omega x^2/\hbar} \tag{3.10}$$

$$\chi_n^+(x) = \mathrm{e}^{+(M\omega/2\hbar)x^2}\chi_n(x) \tag{3.11}$$

and (3.6) itself can be cast in its natural form

$$f_2(x, t|x_0) = \sum \chi_n^+(x_0)\chi_n(x)\mathrm{e}^{-\iota n\omega t} \tag{3.12}$$

and the stationary state is described by

$$f_2(x, t|x_0)|_{\text{stationary}} = \chi_0^+(x_0)\chi_0(x) = \left(\frac{M\omega}{\pi\hbar}\right)^{1/2} \mathrm{e}^{-M\omega x^2/\hbar}. \tag{3.13}$$

It should be noted that the ground state corresponds to the stationary state and the eigenvalue which corresponds to the energy is zero. The density function corresponding to the stationary state is unambiguously given by (3.13) and can be interpreted as the measure arising from modulus measure transformation. At a superficial level it may appear to be in conflict with the Born interpretation according to which the modulus square of the eigenfunction/wavefunction yields the probability density. However, we will demonstrate the viability of a modulus measure in its ability to be in conformity with the Born interpretation. If we use the eigenfunctions of the FP operator, we find

$$\int f_2(x, t|x_0)\chi_n(x_0)\,\mathrm{d}x_0 = \chi_n(x)\mathrm{e}^{-\iota n\omega t} \tag{3.14}$$

$$\int \chi_n^+(x) f_2(x, t|x_0)\,\mathrm{d}x = \chi_n^+(x_0)\,\mathrm{e}^{-\iota n\omega t}. \tag{3.15}$$

Thus we can conclude that the eigenstates for $n \neq 0$ are quasi-stationary states in the strict probabilistic sense and the above equations merely yield the projections of the measure on the individual states. Since quasi-stationary states do not have a unique normalization due to non-conservation of measure, we can take a modulus square measure which in turn is obtained by the corresponding eigenfunction multiplied by its adjoint

$$\chi_n^+(x)\chi_n(x) \tag{3.16}$$

which can be interpreted to be a normalized probability density function. Thus the vacuum is described by

$$\chi_0^+(x)\chi_0(x) = \frac{M\omega}{\pi\hbar}\exp\left(-\frac{M\omega}{\hbar}x^2\right) \tag{3.17}$$

which is the modulus measure of the stationary complex measure density thus bridging the gap in going from the measure density to the corresponding measure density of the quasi-stationary states.

Thus in the CMTF, our approach through the eigenfunctions retains the classical idea that the expected value of the potential energy is half the total energy, the other half coming from the kinetic part. To obtain the momentum distribution, we use the duality concept of de Broglie to identify the Fourier dual (see, for example, Misner *et al* (1972)); thus the Fourier transform $\tilde{\phi}_n(p)$ of the square root of $\chi_n^+(x)\chi_n(x)$ which equals $\phi_n(x)$ plays a dominant role; we have by Plancharel's theorem (Wiener 1933)

$$\int |\phi_n(x)|^2\,\mathrm{d}x = \int |\tilde{\phi}_n(p)|^2 \frac{\mathrm{d}p}{\hbar} \tag{3.18}$$

thus leading to the preservation of the mod square measure. We also have the uncertainty relations preserved in the form of constraints on variances.

Next we proceed to describe the displaced oscillator in the CMTF. The most appropriate way to introduce the displaced oscillator is to specify the drift coefficient; thus if we modify (2.11) and (2.12) by

$$A(x) = -\iota\omega(x-\beta) \qquad B(x) = 2D = \frac{\iota\hbar}{M} \tag{3.19}$$

where β is an arbitrary complex number, we are led to the FP equation

$$\frac{\partial f_2(x,t|x_0)}{\partial t} = +\iota\omega\frac{\partial}{\partial x}(x-\beta)f_2 + \frac{\iota\hbar}{2M}\frac{\partial^2 f_2}{\partial x^2}. \tag{3.20}$$

The above equation can be solved by the method of characteristics:

$$f_2(x,t|x_0) = \left(\frac{M\omega}{\pi\hbar(1-\mathrm{e}^{-2\iota\omega t})}\right)^{1/2}\exp\left(-\frac{M\omega}{\hbar}\frac{[(x-\beta)\,\mathrm{e}^{\iota\omega t}-(x_0-\beta)]^2}{\mathrm{e}^{2\iota\omega t}-1}\right). \tag{3.21}$$

The stationary state solution is the limit as $t \to \infty$ of $f_2(x,t|x_0)$ (under the gimmick $\omega \to \omega - \iota\epsilon$) and is given by

$$\begin{aligned}\lim f_2(x,t|x_0) &= \Pi_\beta(x)\\ &= \left(\frac{M\omega}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{M\omega}{\hbar}(x-\beta)^2\right)\end{aligned} \tag{3.22}$$

which is a genuine complex measure density. We set $M = 1$ to facilitate comparison with conventional results. Now if we take the mod measure and normalize the same, we obtain the probability measure density $\phi_\beta(x)$:

$$\begin{aligned}\phi_\beta(x) &= |\Pi_\beta(x)|\exp\left(-\frac{\omega}{\hbar}(\operatorname{Im}\beta)^2\right)\\ &= \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{2}}\exp\left(-\frac{\omega}{\hbar}(x-\operatorname{Re}\beta)^2\right).\end{aligned} \tag{3.23}$$

If we denote the Fourier transform of $\Pi_\beta^{\frac{1}{2}}(x) = \sqrt{\Pi_\beta(x)}$ by $\tilde{\phi}_\beta^{\frac{1}{2}}(p)$ where

$$\tilde{\phi}_\beta^{\frac{1}{2}}(p) = \frac{1}{\sqrt{2\pi}}\int \Pi_\beta^{\frac{1}{2}}(x)\exp\left(-\iota\frac{px}{\hbar}\right)\mathrm{d}x \tag{3.24}$$

then we find

$$\tilde{\phi}_\beta^{\frac{1}{2}}(p) = \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}}\exp\left(-\frac{1}{2\hbar\omega}[(p+\iota\beta\omega)^2+\omega^2\beta^2+\omega^2(\operatorname{Im}\beta)^2]\right). \tag{3.25}$$

If we now take the mod square we obtain the momentum density function

$$\begin{aligned}|\phi_\beta(p)|\frac{\mathrm{d}p}{\hbar} &= |\tilde{\phi}_\beta^{\frac{1}{2}}(p)|^2\frac{\mathrm{d}p}{\hbar}\\ &= \left(\frac{1}{\pi\hbar\omega}\right)^{\frac{1}{2}}\exp\left(-\frac{(p-\omega\operatorname{Im}\beta)^2}{\hbar\omega}\right)\mathrm{d}p.\end{aligned} \tag{3.26}$$

From (3.23) and (3.26) it follows, using capital letters for the corresponding random variables,

$$\begin{aligned}&\operatorname{Var}X = \frac{\hbar}{2\omega} \qquad \operatorname{Var}P = \frac{\hbar\omega}{2}\\ &\operatorname{Var}X\operatorname{Var}P = \frac{\hbar^2}{4}\end{aligned} \tag{3.27}$$

thus establishing that this corresponds to the wavepacket with minimum variance. We can in fact establish a correspondence to the usual quantum mechanical functions (see, for example, Glauber (1963), Klauder and Sudarshan (1968)):

$$\begin{aligned}&\Pi_\beta^{\frac{1}{2}}(x)\ \text{in CMTF} \to \langle x \mid \beta\rangle \equiv \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(-\frac{\omega}{2\hbar}(x-\beta)^2\right)\\ &\tilde{\phi}_\beta^{\frac{1}{2}}(p)\ \text{in CMTF} \to \langle p \mid \beta\rangle \equiv \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}}\exp\left(-\frac{(p+\iota\beta\omega)^2}{2\hbar\omega}\right).\end{aligned} \tag{3.28}$$

Thus the complex measure density $\Pi_\beta(x)$ taken along with $\tilde{\phi}_\beta(p)$ will achieve all that is required of the coherent state representation (Klauder and Sudarshan 1968). In particular, if the stationary solution $\Pi_\beta(x)$ given by (3.22) is used as the initial condition in the place of (3.2), then the solution of the FP equation (3.1) for any arbitrary time $t > 0$ is given by

$$\begin{aligned}\mathcal{F}_2(x,t) &= \int f_2(x,t|x_0)\left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(-\frac{M\omega}{\hbar}(x_0-\beta)^2\right) \mathrm{d}x_0 \\ &= \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(-\frac{M\omega}{\hbar}(x-\beta\,\mathrm{e}^{-\iota\omega t})^2\right). \qquad (3.29)\end{aligned}$$

Thus the main characteristic of the initial state (minimum variance) is still preserved, the expectation value undergoing a periodic oscillation. It is in this context that we feel the full impact of the CMTF, the modulus measure applied to f_2 ultimately leading to the Born interpretation. The generalized Chebycheff inequality also assumes special importance since it now throws light on the spread about the expected value; the expected value being with reference to the modulus measure now coincides with the conventional expected value or observable expected value.

Having brought the coherent state within the ambit of the CMTF, our next approach is to use the stationary complex measure density to throw light on the distribution of photons in the coherent stream. We note that $\Pi_\beta^{\frac{1}{2}}(x)$ is square integrable and hence belongs to L_2 space; we attempt to obtain the projections on the Hermite functions. Then by the complex extension of the results due to Feller (1971), we can obtain the modulus square of the inner product and identify it as the probability for the process to be in that state. Thus we have by a straightforward calculation

$$\begin{aligned}(\Pi_\beta^{\frac{1}{2}}, \phi_m) &= \int \Pi_\beta^{\frac{1}{2}}(x)\phi_m(x)\,\mathrm{d}x \\ &= \exp\left(-\frac{M\omega}{2\hbar}\left[\frac{\beta^2}{2}+(\mathrm{Im}\,\beta)^2\right]\right)\left(\frac{\beta}{2}\sqrt{\frac{M\omega}{\hbar}}\right)^m \frac{2^{m/2}}{(m!)^{1/2}}. \qquad (3.30)\end{aligned}$$

Thus the probability that the process is in state m is given by

$$|(\Pi_\beta^{\frac{1}{2}}, \phi_m)|^2 = \left(|\beta|^2 \frac{M\omega}{2\hbar}\right)^m \frac{\exp(-(M\omega/2\hbar)|\beta|^2)}{m!}. \qquad (3.31)$$

If we now introduce the usual assumption that quantum energy is $\hbar\omega$ and the electromagnetic field is an assembly of oscillators, then from the CMTF it follows that the stream of photons corresponds to the mth state and the number of photons obey a Poisson distribution with parameter $|\beta|^2\omega/2\hbar$ $(M = 1)$. The above result also brings out the fact that it can be a compound Poisson distribution if β has some structure. The case when β is a function of time is of special importance and corresponds to the problem of interaction of radiation with matter. This is precisely the subject matter under discussion in the next section.

Next we turn our attention to the problem of determination of the density matrix. In the CMTF, it is rather difficult to identify directly from the various possible complex measures. In fact Youssef (1994) has shown that one can handle mixtures of eigenstates by the use of the Bayesian complex probabilities. This is not surprising since the very purpose of generalization of positive definite measures is to accommodate both coherence and mixtures of states. In the past there have been attempts to arrive at a direct probabilistic interpretation for the density matrix in terms of conditional expectation (see Accardi 1981). The starting point is the conditional complex measure density $f_2(x, t|x, t_0)$; we use the standard gimmick of replacing t by $-\iota\beta\hbar$ in the expansion formula (3.12):

$$f_2(x, t|x_0)|_{t=-\iota\beta} = \sum \chi_n^+(x_0)\chi_n(x)\,\mathrm{e}^{-n\beta\hbar\omega}. \tag{3.32}$$

The usual method consists of using the Born interpretation in the expression of χ as defined by (3.9) and setting $x = x_0$ to obtain the required probability. However, in the CMTF, Ψ is a vague quantity and we have to fall back on (3.32). A direct interpretation of (3.32) for $x = x_0$ will lead us nowhere. However, one can continue the same process of reconciliation that was used in passing from the mod measure of f_2 to the mod square measure of the eigenfunctions. Since $\chi_n(x)$ has a direct meaning in the sense that the function multiplied by the adjoint represents the mod square measure density for the nth level, we can delink the initial conditioning and view the summand in (3.32) as a product of $\chi_n(x)$ and its adjoint evaluated at a different point. In such a case we are comfortably placed; the factor $\mathrm{e}^{-n\beta\hbar\omega}$ merely expresses the basic postulate of statistical mechanics that the probability of a system with energy E is proportional to $\mathrm{e}^{-\beta E}$ and $\psi_n^+(x_0)\psi_n(x)$ evaluated at $x = x_0$ denotes the mod square measure density (which in this case is real). In conventional treatments, the zero-point energy is subtracted; however, in the CMTF there is no need to do the same since the ground-state energy is zero. It is our contention that this is perhaps the best way to accommodate the density matrix. However, it should be pointed out that the CMTF is viable enough to deal with mixed states and related problems in any general situation without any explicit reference to a concept such as the density matrix.

Finally, it should be noted that the CMTF is not equivalent to the usual quantum mechanics although it turns out that all the results relating to the harmonic oscillator are in complete agreement with those of quantum mechanics. The diffusion function that enters through the FP equation implies that there is a universal coupling. Second and more fundamentally the uncertainty relations arise from the variance of the relevant dynamical variable and the scale of variance is essentially fixed by the diffusion constant. Even at the level of Schrödinger structure, the FP equation is not equivalent to the Schrödinger equation in view of a multiplying factor which can be interpreted as a gauge transformation. At present the significance of the particular gauge that leads to the Schrödinger equation is rather obscure.

4. Forced harmonic oscillator: matter in interaction with radiation

The forced harmonic oscillator has acquired importance for the simple reason that for many problems in quantum electrodynamics the electromagnetic field can be represented as a set of forced harmonic oscillators. In a sense the complex measure corresponding to a forced harmonic oscillator can be a conditional one, the conditioning being provided by the forcing term which may correspond to matter. It is indeed possible to obtain in an explicit form the measure density corresponding to a forcing term which is a function of time; this perhaps holds the key to the solution of many allied problems. Thus we can consider a harmonic oscillator which is subject to a force $f(t)$. It is assumed that the force is introduced at time $t = t_0$ and we seek the measure density function for such an oscillator at any arbitrary time. The force produces an extra drift which is no doubt time dependent and thus can be calculated purely on classical considerations. Thus we arrive at the drift coefficient $A(x, t)$ now given by

$$A(x, t) = -\iota\omega x + \beta(t) \tag{4.1}$$

$$\beta(t) = \mathrm{e}^{\iota\omega t}\int_0^t f(s)\,\mathrm{e}^{-\iota\omega s}\,\mathrm{d}s. \tag{4.2}$$

The diffusion function $B(x)$ is the same constant function as in (2.11) and the FP equation (2.10) now takes the form

$$\frac{\partial f_2(x,t|x_0,t_0)}{\partial t} = \iota\omega f_2(x,t|x_0,t_0) + [\iota\omega x - \beta(t)]\frac{\partial f_2(x,t|x_0,t_0)}{\partial x} + \frac{\iota\hbar}{2M}\frac{\partial^2 f_2(x,t|x_0,t_0)}{\partial x^2}. \tag{4.3}$$

We can use the transformation

$$z = x\exp\left[\iota\omega(t-t_0) - \int_{t_0}^{t}\beta(s)\,\mathrm{e}^{\iota\omega s}\,\mathrm{d}s\right]$$
$$f_2 = \exp[\iota\omega(t-t_0)]\rho(x,t|x_0,t_0) \tag{4.4}$$

and the initial condition

$$f_2(x,t_0|x_0,t_0) = \delta(x-x_0) \tag{4.5}$$

to solve equation (4.3); thus we obtain

$$\rho = \left(\frac{M\omega}{\pi\hbar(\mathrm{e}^{2\iota\omega t}-\mathrm{e}^{2\iota\omega t_0})}\right)^{1/2}\exp\left(-\frac{M\omega}{\hbar}\frac{[x\,\mathrm{e}^{\iota\omega t} - x_0\,\mathrm{e}^{\iota\omega t_0} - \int_{t_0}^{t}\beta(s)\,\mathrm{e}^{\iota\omega s}\,\mathrm{d}s]^2}{\mathrm{e}^{2\iota\omega t}-\mathrm{e}^{2\iota\omega t_0}}\right). \tag{4.6}$$

Based on solution (4.6), the connection to the Feynman path integral formula has been established in paper II. We shall not discuss this aspect any further. For notational convenience we choose $t_0 = 0$ and write the solution for the conditional measure density f_2 as

$$f_2(x,t|x_0) = \left(\frac{M\omega\,\mathrm{e}^{2\iota\omega t}}{\pi\hbar(\mathrm{e}^{2\iota\omega t}-1)}\right)^{1/2}\exp\left(-\frac{M\omega}{\hbar(\mathrm{e}^{2\iota\omega t}-1)}\left\{x\,\mathrm{e}^{\iota\omega t} - x_0 - \int_0^t\beta(s)\,\mathrm{e}^{\iota\omega s}\,\mathrm{d}s\right\}^2\right). \tag{4.7}$$

We expand f_2 indirectly by seeking the projection of f_2 on $\chi_m(x_0)$:

$$\int f_2\chi_m(x_0)\,\mathrm{d}x_0 = \chi_m\left(x - \int_0^t\beta(\xi)\,\mathrm{e}^{-\iota\omega(t-\xi)}\,\mathrm{d}\xi\right)\mathrm{e}^{-\iota m\omega t}. \tag{4.8}$$

Thus we have

$$f_2 = \sum_m \chi_m^+(x_0)\chi_m\left(x - \int_0^t\beta(\xi)\,\mathrm{e}^{-\iota\omega(t-\xi)}\,\mathrm{d}\xi\right)\mathrm{e}^{-\iota m\omega t}. \tag{4.9}$$

From this we can deduce the dual relation

$$\int f_2\chi_n^+\left(x - \int_0^t\beta(\xi)\,\mathrm{e}^{-\iota\omega(t-\xi)}\,\mathrm{d}\xi\right) = \chi_n^+(x_0)\,\mathrm{e}^{-\iota n\omega t y}. \tag{4.10}$$

Equation (4.8) for $m = 0$ takes the form

$$\int f_2\chi_0(x_0)\,\mathrm{d}x_0 = \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left(-\frac{M\omega}{\hbar}\left(x - \int_0^t\beta(\xi)\exp(-\iota\omega(t-\xi)\,\mathrm{d}\xi\right)^2\right). \tag{4.11}$$

To obtain the probability we take the adjoint of χ_0 and multiply it by the right-hand side of (4.8) (mod square measure); it is worth noting that the same result can be obtained in the modulus measure framework by dealing with $\Pi_{gd}(x,t)$, the measure density conditional upon the initial state at $t = 0$ being the ground state; in this case we have

$$\Pi_{gd}(x,t) = \int f_2\Pi(x)\,\mathrm{d}x \tag{4.12}$$

where $\Pi(x)$ is the measure density of the ground state given by (3.4). On evaluation of the integral, we find

$$\Pi_{gd}(x,t) = \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{2}} \exp\left(-\frac{M\omega}{\hbar}\left(x - \int_0^t \beta(s)\exp(-\iota\omega(t-s)\,\mathrm{d}s\right)^2\right) \tag{4.13}$$

a result different from (3.5) as is to be expected. In the mod measure framework, this leads to the same result as that obtained from (4.11).

Next we expand $\Pi_{gd}^{\frac{1}{2}}(x,t)$ in terms of an orthonormal system in L_2 to which space $\Pi_{gd}^{\frac{1}{2}}$ legitimately belongs. We choose the basis as the Hermite functions ϕ_m. Set

$$\alpha = \int_0^t \beta(s)\,\mathrm{e}^{-\iota\omega(t-s)}\,\mathrm{d}s \tag{4.14}$$

where α is real if f is real and will have a non-vanishing imaginary part if f is complex valued. Thus we find the projections given by

$$\begin{aligned}(\Pi_{gd}^{\frac{1}{2}}, \phi_m) &= \int \Pi_{gd}^{\frac{1}{2}}\phi_m(x)\,\mathrm{d}x \\ &= \exp\left(-\frac{M\omega}{2\hbar}\left[\frac{\alpha^2}{2} + (\mathrm{Im}\,\alpha)^2\right]\left(\frac{\alpha}{2}\sqrt{\frac{M\omega}{\hbar}}\right)^m\left(\frac{2^m}{m!}\right)^{\frac{1}{2}}\right)\end{aligned} \tag{4.15}$$

so that the probability that the corresponding process is in state m is given by

$$|(\Pi_{gd}^{\frac{1}{2}}, \phi_m)|^2 = \frac{((|\alpha|)^2 M\omega/2\hbar)^m}{m!}\exp\left(-\frac{M\omega}{2\hbar}|\alpha|^2\right). \tag{4.16}$$

The above result reconfirms the earlier result in section 3 that if α does not have any structure then the resulting stream corresponds to a coherent beam of photons. In a more general situation, (4.16) represents a conditional probability thereby making it transparent that the process is a compound Poisson process. The above result can be the basis for the description of light amplification for α and can be the structure function for the atomic system of two-level type.

Now we are comfortably placed to discuss the problem of radiation in interaction with matter. To proceed further, we note that we need to follow closely the formalism of FH (ch 9) where Maxwell's equations are reduced to forced harmonic oscillators:

$$\ddot{a}_{1k} + k^2c^2a_{1k} = \sqrt{4\pi}\,j_{1k}^* \tag{4.17}$$

with a similar equation for a_{2k} where a_{1k} and a_{2k} are the two transverse Fourier components of the vector potential. To this we add the hypothesis of quantum electrodynamics that the oscillators defined above are quantum oscillators; the current components j_{1k}, j_{2k} arise from the matter term. Thus for each of the modes the forcing term is identified as $\sqrt{4\pi}\,j_{1k}^*$ and in the CMTF we are interested in the conditional measure density corresponding to the initial state being the ground state. Furthermore, if we constrain the final state also to be the ground state we can proceed to estimate the effect in the level shift. Thus the ground-state measure density (for one of the modes) or rather its square root is given by $(\Pi_{gd}^{\frac{1}{2}}, \phi_0)\phi_0$; since we are working in the momentum representation we take the Fourier transform so that the desired measure is

$$(\Pi_{gd}^{\frac{1}{2}}, \phi_0)\tilde{\phi}_0 = \left(\frac{\pi\hbar}{M\omega}\right)^{1/4}\exp\left(-\frac{p^2}{2M\omega\hbar}\right)(\Pi_{gd}^{\frac{1}{2}}, \phi_0). \tag{4.18}$$

We can follow Feynman's method as given in section 9.4 of FH; however, there is a difference. We first deal with the matter part and use the perturbation methods. In the final result we take the modulus square to account for the correct probability measure. It is shown in the appendix that we are led to the formula 9.68 with the modification that there is an extra multiplying factor $\exp(-|k|/\hbar c)$ with the normalization constant in the integrand for each of the modes. Thus the Lamb shift calculations in section 9.6 go through with this modification and the cut-off is justified as a valid numerical approximation. Exactly the same considerations apply for the calculation using the relativistic Dirac wavefunction and states. Thus in the CMTF there are no difficulties nor divergences of the type encountered earlier.

It is not difficult to explain why such a simplicity arises in the CMTF. The arguments should not be construed as criticism of the method of evaluation; rather it is the weakness of the Hilbert space approach in which the role of probability is purely artificial. For instance, if we follow the arguments for the ground-state energy calculation, the probability of transition of vacuum to vacuum is obtained by evaluating a scalar product and we are then left with the predicament of dealing with the modes over which an integration has already been performed in the course of closing the bracket. On the other hand, the CMTF consistently avoids such a situation and a measure theoretic interpretation is possible at every stage in the calculation.

5. Summary and conclusions

In this paper we have discussed the possibility of describing quantum phenomena by enlarging the elementary notions of probability based on the positive definite measure. Instead of interpreting the two-slit experiment as a symptom of failure of classical notions and arriving at a new structure, we wish to explore the idea that the results of the two-slit experiment can be accommodated within the framework of an extended measure. It was with this motive that complex measurable processes were introduced; the evolution of the complex measure density, particularly of the Markov type, brings to the fore the diffractive form while the very nature of the complex measure describes the interference phenomenon. It is found that for describing particles with structures we have to go beyond the framework of the complex measure with an extended measure of quaternion type; a simple model of a two-valued process within such a framework fused with a complex measure structure yields the Dirac equation in the Weyl representation. It is rather difficult to relate results in the CMTF directly to those in the traditional quantum mechanics; however, it is possible to make connection to the PIF developed by Feynman. In a sense, the path integral serves as a nice intermediary between CMTF and the conventional formalism of quantum mechanics. It is with this in view that we have attempted to provide a comprehensive description of the harmonic oscillator in all its aspects. Some results and connections have already been provided earlier (Srinivasan 1995); in this paper we have shown how topics such as quasi-stationary states and coherent states can be handled within the framework of extended measures and measurable processes. In order to make the results amenable to statistical inference through frequency ratios, it is necessary to make a measure transformation leading to a positive definite measure. It turns out that the modulus measure proposed earlier is eminently suited and we show how this can be reconciled with the usual Born interpretation. Since conditioning is one important feature from a probabilistic point of view, we have discussed this extensively in the hope that in the near future serious topics such as the collapse of a wavepacket can be brought within the ambit of the present approach. We have taken the opportunity to discuss the forced harmonic oscillator in a rather detailed way in the final section; we have also provided a detailed connection to the results of Feynman and Hibbs relating to the Lamb shift calculations. It turns out that, if the CMTF is adopted, there are no divergence difficulties and the cut-off is justified as a valid numerical approximation. Finally, we note that the extended measure approach can be further extended to include Grassman algebraic structures; this gives us hope that one day Fermi oscillators can be discussed in as versatile a manner as the harmonic oscillators and that an inroad can be made into the modern gauge theory.

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Appendix

The starting point is the formula (4.15) and we are dealing with the case when both the initial and final state of the radiation correspond to the ground state. In taking over the results, we note that the factor $\exp -(M\omega/\hbar)(\operatorname{Im}\alpha)^2$ is inserted to ensure that modulus measure transformation when effected leads to the properly normalized momentum measure density. On the other hand, if we remove the factor we obtain

$$(\pi_{gd}^{\frac{1}{2}}, \phi_m(x)) = \frac{(\alpha\sqrt{M\omega/2\hbar})^m}{(m!)^{\frac{1}{2}}} \exp\left(-\frac{M\omega}{4\hbar}\alpha^2\right) \tag{A.1}$$

so that the square leads to a properly normalized Poisson distribution in a complex measure theoretic sense. Incidentally this demonstrates how the complex measure can be carried forward and retained up to this level. The parameter α is given by

$$\begin{aligned}\alpha &= \int_0^t \beta(s)\,\mathrm{e}^{-\iota\omega(t-s)}\,\mathrm{d}s \\ &= \frac{1}{\omega}\int_0^t f(u)\sin\omega(t-u)\,\mathrm{d}u\end{aligned} \tag{A.2}$$

where the forcing term is taken to correspond to the current in the reduced Maxwell equations (4.17) and thus we have for any particular mode

$$f(t) \to \sqrt{4\pi}\, j^*(t). \tag{A.3}$$

We are primarily interested in the radiation going over from ground to ground state and this is given by the conditional ground-state measure density ($m = 0$); since we are working in the momentum representation, the relevant measure is given by

$$|(\pi_{gd}^{\frac{1}{2}}, \phi_0)|^2 \tilde{\phi}_0^2 = \exp\left[-\frac{p^2}{M\omega\hbar} - \frac{M\omega}{2\hbar}|\alpha|^2\right]. \tag{A.4}$$

Using A.2 and A.3 we find

$$\begin{aligned}|\alpha|^2 &= \frac{4\pi}{\omega^2}\int_0^t\int_0^t j^*(s)\,j(u)\sin\omega(t-s)\sin\omega(t-u)\,\mathrm{d}s\,\mathrm{d}u \\ &= I_1 + I_2 + I_3 + I_4\end{aligned} \tag{A.5}$$

where

$$I_1 = \frac{\pi}{4\omega^2}\int_0^t\int_0^t j(s)j^*(u)\,\mathrm{e}^{-\iota\omega(s-u)}\,\mathrm{d}s\,\mathrm{d}u \tag{A.6}$$

$$I_2 = \frac{\pi}{4\omega^2}\int_0^t\int_0^t j^*(s)j(u)\,\mathrm{e}^{\iota\omega(s-u)}\,\mathrm{d}s\,\mathrm{d}u \tag{A.7}$$

$$I_3 = \frac{\pi}{4\omega^2}\int_0^t\int_0^t j^*(s)j(u)\,\mathrm{e}^{\iota\omega(t-s)+\iota\omega(t-u)}\,\mathrm{d}s\,\mathrm{d}u \tag{A.8}$$

$$I_4 = \frac{\pi}{4\omega^2}\int_0^t\int_0^t j^*(s)j(u)\,\mathrm{e}^{-\iota\omega(t-s)-\iota\omega(t-u)}\,\mathrm{d}s\,\mathrm{d}u. \tag{A.9}$$

By straightforward manipulation we have

$$I_1 + I_2 = \frac{\pi}{\omega^2}\int_0^t\int_0^t j^*(s)j(u)[\mathrm{e}^{-\iota\omega|s-u|} + \mathrm{e}^{\iota\omega|s-u|}]\,\mathrm{d}s\,\mathrm{d}u. \tag{A.10}$$

It is to be especially noted that apart from the conditioning due to the initial ground state of the radiation a further conditioning arises from the matter system through the matter current entering into the picture as a forcing term; thus we have to average over the matter system and this is done by taking an expectation value over matter (since the conditional measure density is just a random variable).

We note that in the calculation of Lamb shift electromagnetic radiation is treated essentially as a perturbation. The discussion in the last paragraph of section 2 shows that the atomic levels can be entracted since the FP equation leads to a Schrödinger structure; the conditional measure as given by (A.4) needs to be averaged over the matter system treating it as a perturbation. We can adapt the results of ch 6 of FH relating to a time-dependent perturbation. In this process as explained earlier the CMTF maintains its identity with its versatility of being a measure. Thus we need to take the expectation value of the right-hand side of A.4 treating $|\alpha|^2$ as a perturbation: there is a little bookkeeping, $M = 1$ and $p \mapsto k = \omega/c$.

To zeroth order (when α is neglected) the expectation value is

$$\left[\exp -\iota\frac{E_M t}{\hbar}\right]\exp -\frac{p^2}{\hbar\omega}$$

which is the analogue of equation (9.66) of FH. However, there is a difference: the expectation value is still a measure (density) in momentum and we need to integrate over momentum since we are in the process of averaging over matter only. Next we look at first order; the contribution from $I_1 + I_2$ given by (A.10) effectively comes from the first term (containing the factor $\exp(-\iota\omega|s-u|)$ in view of the overall energy conservation; the contribution from I_3 goes to zero. $\omega \to \omega + \iota\epsilon$ while I_4 gives a contribution to second order.

Taking care of the fact that the expression given by (A.5) is for a particular mode and that we need to integrate the measure density over the momentum, we find the contribution to first order (in the notation of FH) is

$$\lambda'_{MM} = -\frac{\iota}{\hbar}(\Delta E)T\,\mathrm{e}^{-\iota E_M t/\hbar} \tag{A.11}$$

$$\Delta E = 4\pi\hbar\sum_N\int[|(j_{1k})_{NM}|^2 + |(j_{2k})_{NM}|^2]\frac{\mathrm{e}^{-|k|/\hbar c}}{(2\pi)^3 2kc(E_M - E_N - k\hbar c + \iota\epsilon)}\,\mathrm{d}^3k \tag{A.12}$$

which is essentially formula (9.68) of FH except for the multiplying factor $\mathrm{e}^{-|k|/\hbar c}$ and a normalization constant. Thus from the CMTF point of view we can assert that the cut-off employed by Feynman can be visualized as a numerical approximation. It is very likely that there is a difference in the actual level shift when the factor $e^{-|k|/\hbar c}$ is used, the advantage being the disappearance of divergences at any level of calculation.

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